The Spatial Viscous Instability of a Two-Dimensional Developing Mixing Layer

Taewon Seo*

(Received October 27, 1993)

This paper deals with the hydrodynamic instability of the free shear layer. The development of mixing layers at downstream of a splitter plate is initially dominated by a linear instability mechanism. The linear instability problem is an eigenvalue problem and solved by using the shooting method with orthonormalization to maintain linear independence of the solutions. Bi-directional integration with matching at the intermediate point within the infinite domain was used to ensure the convergence and the accuracy in satisfying the boundary conditions at $\pm\infty$. It was found that when the Reynolds number increases, the amplification rate approaches to that of the inviscid solutions.

Key Words: Viscous Instability, Bi-Directional Integration, Matching Point Orthonormalization, Shear Layer

Nomenclature

$C_{1,2}$: The complex coefficients at $-\infty$
c : The phase speed
$D_{1,2}$: The complex coefficients at ∞
P : Pressure
$\tilde{p}$: The large-scale structure of pressure
$\hat{p}$: The eigenfunction of pressure
R : Velocity ratio
Re : Reynolds number
t : Time
$U_{\pm\infty}$: The free stream velocities
$\tilde{u}, \tilde{v}$: Large-scale structtures
$\hat{u}, \hat{v}$: The eigenfunction of streamwise and cross-streamwise velocities
α : Complex eigenvalue
β : Frequency parameter
δ : Maximum slope thickness
υ : Dynamic viscosity
ρ : Fluid density

1. Introduction

This paper is concerned with the hydrodynamic instability of the free shear layer. The development of mixing layers downstream of a splitter plate is initially dominated by a linear instability mechanism. A great deal of work has been done on this subject and for general review of the hydrodynamic stability of parallel flows such as the free shear layer the readers should consult by Drazin and Reid(1981). Gaster(1962) clearly showed that spatially developing disturbances better represent the stability characteristics of a developing shear layer. Michalke(1965) first calculated the spatial stability of a parallel shear layer. His results showed the amplification rates for different frequencies for an inviscid shear layer with the mean velocity profile approximated by a hyperbolic tangent. Morris(1976) studied the stability of three axisymmetric jet profiles which represented the flow field of an incompressible jet. Monkewitz and Huerre(1982) have studied the theoretical dependence of spatially growing waves on free-stream velocity ratio for the hyperbolic tangent profile. Seo(1993) first investigated the eigenvalues and eigenfunctions of the stability for unbounded viscous shear layer by solving the eigenvalue problem with a shooting solution matching method instead of one-directional integration method.

* R and D Center, Korea Heavy Industries and Constructions Co. Ltd 555 Guygok-Dong, Changwon, Kyungnam 641-792, Korea

Hydrodynamic instability has been recognized as one of the fundamental mechanism to understand the transition from laminar to turbulent flow. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. It has many applications in engineering, in meteorology and oceanography, and in astrophysics.

Our interest in this study lies in obtaining solutions for the viscous spatial two-dimensional stability corresponding to the classical self-similar velocity profile of the free shear layer. The infinite domain combined the effects of the viscosity makes the numerical solution of the viscous linear stability problem quite challenging.

2. Mathematical Formulation of the Linear Stability Analysis

We assume that the flow is perturbed and use the tilde to indicate the perturbation. Use the nondimensional parameters

$$x \rightarrow \frac{x}{\delta} \tag{1}$$

where δ represents the maximum slope thickness.

Thus

$$\begin{aligned} u &= U(\eta)+\tilde{u}(x,\eta,t) \\ v &= \tilde{v}(x,\eta,t) \\ p &= P(x)+\tilde{p}(x,\eta,t) \end{aligned} \tag{2}$$

where the mean velocity $U(\eta)=1-R\tanh(\eta)$ with $R=\dfrac{(U_\infty-U_\infty)}{(U_\infty+U_\infty)}$

the velocity ratio of the shear layer.

If Eq. (2) is introduced into the Navier-Stokes equations and linearized the equations by assuming the fluctuations and their derivatives have small amplitudes, we get

$$\begin{aligned} &\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}=0 \\ &\frac{\partial\tilde{u}}{\partial t}+U\frac{\partial\tilde{u}}{\partial x}+\frac{\partial U}{\partial\eta}\tilde{v} \\ &=-\frac{\partial\tilde{p}}{\partial x}+\nu\left(\frac{\partial^2\tilde{u}}{\partial x^2}+\frac{\partial^2\tilde{u}}{\partial\eta^2}\right) \\ &\frac{\partial\tilde{v}}{\partial t}+U\frac{\partial\tilde{v}}{\partial x} \\ &=-\frac{\partial\tilde{p}}{\partial\eta}+\nu\left(\frac{\partial^2\tilde{v}}{\partial x^2}+\frac{\partial^2\tilde{v}}{\partial\eta^2}\right). \end{aligned} \tag{3}$$

The boundary conditions are

$$\eta\rightarrow\pm\infty:\ \tilde{u},\ \tilde{v},\ \tilde{p}\rightarrow 0. \tag{4}$$

Assume that the solutions of Eq. (3) are

$$\begin{bmatrix}\tilde{u}(x,\eta,t)\\ \tilde{v}(x,\eta,t)\\ \tilde{p}(x,\eta,t)\end{bmatrix}=\begin{bmatrix}\hat{u}(\eta)\\ \hat{v}(\eta)\\ \hat{p}(\eta)\end{bmatrix}e^{i\alpha(x-ct)}+c.c. \tag{5}$$

where $c.c.$ is the complex conjugate. α represents a complex wave number and c a phase speed. $\hat{u}$, $\hat{v}$ and $\hat{p}$ are eigenfunctions to be calculated by linear stability analysis.

When we substitute Eqs. (5) into (3), we get

$$\begin{aligned} &i\alpha\hat{u}+\frac{d\hat{v}}{d\eta}=0 \\ &i\alpha(U-\beta/\alpha)\hat{u}+\frac{dU}{d\eta}\hat{v} \\ &=-i\alpha\hat{p}+\frac{1}{Re}\left(\frac{d^2\hat{u}}{d\eta^2}-\alpha^2\hat{u}\right) \\ &i(U-\beta/\alpha)\hat{v} \\ &=-\frac{d\hat{p}}{d\eta}+\frac{1}{Re}\left(\frac{d^2\hat{u}}{d\eta^2}-\alpha^2\hat{u}\right). \end{aligned} \tag{6}$$

The imaginary part of the wave number $\alpha(=\alpha_r+i\alpha_i)$ determines the stability of the flow; the flow is stable if α_i has positive value, neutral if α_i is equal to zero, and unstable if α_i has negative value. $\hat{u}$, $\hat{v}$ and $\hat{p}$ are complex amplitude functions.

If we use $D\hat{u}\equiv\dfrac{d\hat{u}}{d\eta}$ and if we define the variable vector V as

$$V=\begin{bmatrix}\hat{u}\\ D\hat{u}\\ \hat{v}\\ \hat{p}\end{bmatrix} \tag{7}$$

then Eq. (6) become

$$\frac{dV}{d\eta}=MV \tag{8}$$

where M is 4×4 matrix as followed;

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \gamma & 0 & Re\frac{dU}{d\eta} & iaRe \\ -ia & 0 & 0 & 0 \\ 0 & -\frac{ia}{Re} & \frac{-\gamma}{Re} & 0 \end{bmatrix}$$

$$\text{where } \gamma = a^2 + iaRe(U - \frac{\beta}{a}). \qquad (9)$$

The system of ordinary differential Eq. (8) together with the boundary conditions given by Eq. (4) poses eigenvalue problem with as the eigenvalue and $\hat{u}(\eta)$, $\hat{v}(\eta)$ and $\hat{p}(\eta)$ as the eigenfunctions that will be solved by a numerical method.

3. Numerical Method for the Problem

In the numerical solution of the eigenvalue problem on infinite intervals, a common method of procedure is to replace the infinite interval, $(-\infty, \infty)$, by a finite one. The main problem then is to determine the appropriate boundary conditions to be imposed at a finite interval (say $\eta = \pm \eta_\infty$).

To get the asymptotic solutions, the mean velocity is constant at $\eta = \pm \eta_\infty$. From Eq. (8), we will get the asymptotic solutions.

As $\eta \to -\infty$ the asymptotic solutions becomes

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{bmatrix} = C_1 \begin{bmatrix} i \\ 1 \\ -i(U_{-\infty} - \beta/a) \end{bmatrix} e^{a\eta} + C_2 \begin{bmatrix} i\chi/a \\ 1 \\ 0 \end{bmatrix} e^{a\chi} \qquad (10)$$

or

$$V_- = C_1 V_{-1} + C_2 V_{-2} \qquad (11)$$

and at $\eta \to \infty$ the asymptotic solutions are

$$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{bmatrix} = D_1 \begin{bmatrix} -i \\ 1 \\ i(U_{+\infty} - \beta/a) \end{bmatrix} e^{-a\eta} + D_2 \begin{bmatrix} -i\chi/a \\ 1 \\ 0 \end{bmatrix} e^{-\chi\eta} \qquad (12)$$

or

$$V_+ = D_1 V_1 + D_2 V_2 \qquad (13)$$

where $\chi = \sqrt{a^2 + iaRe(U_{\pm\infty} - \beta/a)}$, $V_{\pm 1}$ is the inviscid solution and $V_{\pm 2}$ is the viscous solution.

These asymptotic solutions are used as initial conditions to solve the system of Eq. (8).

Equation (8) is integrated from $-\eta$ to 0 and from η to 0, and two computed results are matched at the matching point (in here matching point is 0). It is found that this matching method gives more accurate results than one-directional integration method.

A Runge-Kutta-Fehlberg method is used to solve the ordinary differential equations. This scheme controls the step size by keeping the estimate of the local error below the user specified tolerance (in this study 10^{-08}). The estimate of local error is obtained by comparing the two values evaluated by a fourth-order method and a fifth-order method.

With correct eigenvalue, the computed solutions at the matching point satisfy

$$C_1 V_{-1} + C_2 V_{-2} = D_1 V_1 + D_2 V_2. \qquad (14)$$

Rewriting the Eq. (14) in a matrix form,

$$GC = 0. \qquad (15)$$

The eigenvalue is determined if the determinant of the matrix G is equal to zero ;

$$\Delta \equiv det(G) = 0. \qquad (16)$$

This eigenvalue is calculated using the iterative technique.

If the Reynolds number is higher, the independent solutions lose their linear independence because of contaminating error particularly in the viscous region of the mixing layer. Therefore, the eigenvalue and eigenfunctions obtained by the superposition method are not accurate at all. An orthonormalization method is used in order to keep the independence of these sets of solutions. The orthonormalization method was developed by Conte(1966) and Davey(1973) and has been successfully applied by Morris(1976) in axisymmetric jet and by Seo(1993) in a plane mixing layer.

4. Results and Discussion

The computed two-dimensional spatial amplification rates for the hyperbolic tangent profile with velocity ratio $R = 0.31$ are plotted in Fig. 1

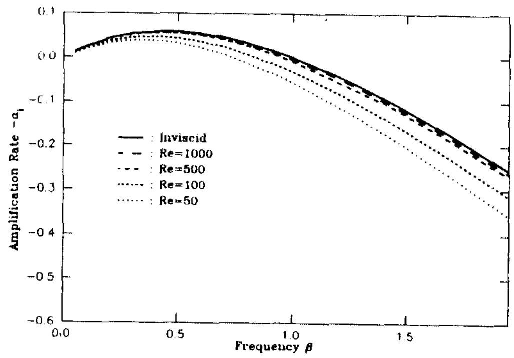

Fig. 1 Amplification rates $-\alpha_i$ versus nondimensional frequency for the case of two-dimensional disturbance with several Reynolds numbers

for various Reynolds numbers. The inviscid solution is also shown. The latter was obtained numerically by integrating the second-order inviscid stability equation (Ragleigh equation) with the appropriate choice of integration contour to accommodate the singularity at $U=c$(Liu and Nikitopoulos, 1987). The amplication rate increases as the Reynolds number increases at fixed frequency β. When the Reynolds number is 1,000, the amplification rates are almost equal to those obtained by the inviscid analysis. For higher frequencies beyond the neutral value, the shear layer becomes stable and the disturbance are damped. As long as the amplification rate $-\alpha_i$ is positive, the large-scale disturbances are amplified. When α_i becomes zero, the disturbance is neither amplified nor damped and is neutrally stable. From Fig. 1 we see that the neutral stability point is moved to lower frequencies as the Reynolds number is increased because of the damping effect of viscosity. In Fig. 2 we have calculated the critical Reynolds number to be approximately 12.5. In region I the two-dimensional disturbance is unstable and stable in region II.

In Fig. 3 the eigenfunctions $\hat{u}$, $\hat{v}$ and $\hat{p}$ for the two-dimensional disturbance are shown near the most amplified frequency for two Reynolds numbers 50, and 1,000. The trends in the amplified region are consistent with those of the experiments by Gaster, Kit and Wygnanski(1985) and Weisbrot and Wygnanski(1988). In Fig. 4, we show the Reynolds stresses induced by the large-scale structures for $Re=1,000$. The magnitudes of the normal stresses decay slowly with distance away from the center of the mixing layer. The maxima of Reynolds stress occur around the center of the shear layer and the trend of our results shows good agreement with the experimen-

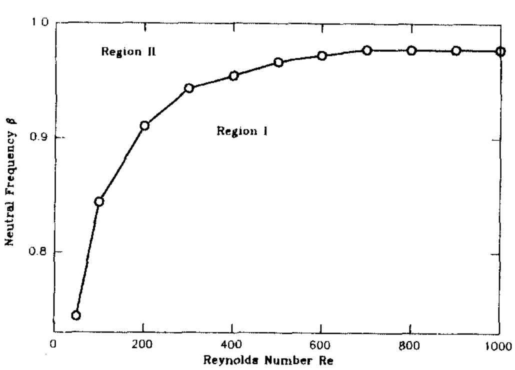

Fig. 2 The neutral stability curve for two-dimensional disturbance

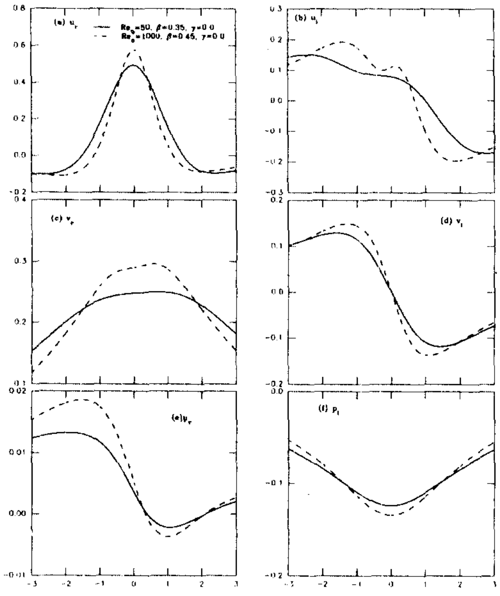

Fig. 3 The eigenfunction near the most amplified frequency $\beta_{max}=0.45$ for Re$=1,000$ and 0.35 for Re$=50$

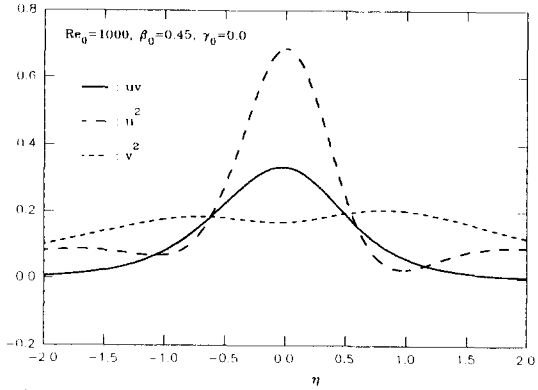

Fig. 4 The large-scale Reynolds stress for Re=1,000 and $\beta_{max}=0.45$

tal observation by Weisbrot and Wygnanski(1988). In the amplified frequency region the Reynolds stress induced by the large-scale structure is positive. A positive Reynolds stress implies that the energy of the mean flow will transfer to the large-scale structure.

5. Conclusions

The eigenvalue and the eigenfunctions of a spatially developing two-dimensional shear layer was calculated using local linear stability theory. It was found that bi-directional integration method is more efficient and accurate than the one-directional method. If the amplification rate $-\alpha_i$ is positive, the flow is amplified. As the Reynolds number increases, the amplification rate approaches to that of the inviscid solutions. In order to verify the used numenrical method, the computed eigenvalues at high Reynolds number have been successfully compared with those of the inviscid numerical results done by Michalki(1965).

References

Conte, S. D., 1966, "The Numerical Solution of Linear Boundary Value Problems," *SIAM Review* 8, pp. 309~321.

Davey, A., 1973, "A Simple Numrical Method for Solving Orr-Sommerfeld Problem," *Quart. J. Appl. Math.*, Vol. 26, pp. 401~441.

Drazin, P. G. and Reid, W. H., 1981, "Hydrodynamic Stability," Cambridge University Press.

Gaster, M., 1962, "A Note on the Relation Between Temporally-Increasing and Spatially-Increasing Disturbances in Hydrodynamic Stability," *J. Fluid Mech.*, Vol. 14, pp. 222~224.

Gaster, M., Kit, E. and Wygnanski, I., 1985, "Large-scale Structures in a Forced Turbulent Mixing Layer," *J. Fluid Mech.*, Vol. 150, pp.23~39.

Liu, J. T. C. and Nikitopoulos, D. E., 1987, "Nonlinear Binary-Mode Interactions, in a Developing Mixing Layer," *J. Fluid Mech.*, Vol. 179, pp. 345~370.

Michalki, A., 1965, "On Spatially Growing Disturbances in a Inviscid Shear Layer," *J. Fluid Mech.*, Vol. 23, pp. 521~544.

Morris, P. J., 1976, "The Spatial Viscous Instability of Axisymmetric Jets," *J. Fluid Mech.*, Vol. 77, pp. 511~529.

Seo, Taewon., 1993, "Three-Dimensional Large-Scale Coherent Structures in a Developing Plane Mixing Layer,", Ph. D Thesis, University of Louisiana State, USA.

Weisbrot, I. and Wygnanski, I., 1988, "On Coherent Structures in a Highly Exicited Mixing Layer," *J. Fluid Mech.*, Vol. 195, pp. 137~159.